Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Congress, they were invited to a sumptuous entertainment at Potsdam by their Royal Highnesses the Crown Prince and Princess, and to a magnificent supper given by some members of the municipality of Berlin, though styling themselves modestly only a few friends of statistics. His Excellency the Prime Minister, Von Bismark, and His Excellency Count Eulenbourg, Minister of the Interior, invited the official members to dinners at their private residences; and to the courtesy and unvarying attention of Dr. Engel, and other Prussian authorities and private friends, is owing a debt of grateful remembrance not easy to be repaid nor soon to be forgotten.

*Cotton-Spinning Problem.** *By* W. S. B. WOOLHOUSE, F.R.A.S.

SOME years ago, before the enactment of the Ten Hours Factory Bill, I had the honour to be engaged by the Earl of Shaftesbury (then Lord Ashley) on an investigation of the average distances traversed daily by children and others employed as "piecers" in the cotton mills about Manchester and the surrounding districts. After taking the various dimensions of the mules or spinning-jennies, and making the requisite observations as to the number of strokes of each machine per minute, and the average number of breakages of the threads at each stroke, I found that the necessary calculation involved a preliminary discussion of a somewhat curious mathematical problem.

As the subject, in its peculiar connexion with the theory of probabilities, is both novel and interesting, it has been suggested to me that it might be acceptable to give a mathematical enunciation of the problem, with a sketch of the method of solution I then adopted. The resulting formula is elegant, and easy of calculation, and its practical application to the important inquiry then under consideration showed that in many cases the distances travelled exceeded thirty miles per day, a task that must have been rendered additionally irksome and fatiguing by the continual stooping posture.

To comprehend the subject in a mathematical point of view, the problem presented for investigation may be stated as follows:—

PROBLEM.—Supposing n points to be taken promiscuously on a line of a given length, and that a person stationed somewhere on the line is required to proceed to all the points by the shortest route, determine the average distance that he may be expected to travel.

---

* As the production of an eminent actuary, this curious and interesting problem will, no doubt, be acceptable to our readers.—ED. *A. M.*

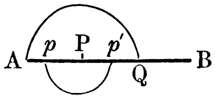
The number of ways in which one point can be taken in a line of a given length must evidently be proportional to that length, and as we are about to deal with proportions only we may represent the number by the length. Thus x will represent the number of ways in which one point can appear in a line x; also, the absolute number of positions being indefinitely great, a little consideration will show that the number of ways in which

$$\text{two points can appear} = \int x dx = \frac{x^2}{2},$$

$$\text{three} \quad ,, \qquad ,, \quad = \int \frac{x^2}{2} dx = \frac{x^3}{2.3},$$

$$\&\text{c.} \qquad\qquad \&\text{c.}$$

It will hence be found that the number of ways for $n-1$ points $= \frac{x^{n-1}}{1.2.3 \ldots . n-1}$, and for n points $= \frac{x^n}{1.2.3 \ldots . n}$.



Let $AB = a$ be the given line, and, P being the position of the person, or piecer, let $AP = x$, $PB = y$. Also let the average distances from P to the two extreme points be thus denoted:

$$\left.\begin{matrix}\Delta \\ \delta\end{matrix}\right\} \text{ to the } \left\{\begin{matrix}\text{remote} \\ \text{nearer}\end{matrix}\right\} \text{ extreme point,}$$

provided the points occur on the same side of P;
and Δ', δ the same whenever the points happen to be distributed on both sides of P.

These contingent distances Δ, δ, Δ', δ', which may severally occur on either side of P, we shall consider seriatim.

First, with regard to Δ, suppose the points to fall between P and A, the most remote point p being at a distance $pP = \beta$. The remotest point being at p, the remaining $n-1$ points must evidently fall between p and P, and we have found that the number of ways in which this can happen $= \frac{\beta^{n-1}}{1.2.3 \ldots n-1}$. Divide this by the total number of ways for the n points in the whole line, viz., $\frac{a^n}{1.2.3 \ldots n}$, and we get the probability of $n-1$ points falling between p and $P = \frac{n(\beta)^{n-1}}{a^n}$. The value of the liability of having to traverse the distance β, subject to this chance, is, therefore, $\frac{n(\beta)^{n-1}}{a^n} \beta$; and, as regards the possibility of the points falling wholly on AP, the value of Δ will be $\int_0^x \frac{n(\beta)^{n-1}}{a^n} \beta d\beta$.

Similarly the value of Δ in respect of the possibility of the points falling wholly on PB will be $\int_0^y \frac{n(\beta)^{n-1}}{a^n}\beta d\beta$.

The total value of Δ is therefore

$$\Delta=n\int_0^x \frac{(\beta)^{n-1}}{a^n}\beta d\beta+n\int_0^y \frac{(\beta)^{n-1}}{a^n}\beta d\beta.$$

Next, with respect to the distance δ, let $p\text{P}=\beta$ be the distance of the nearer extreme point from P when the n points are on AP; then the remaining $n-1$ points must be found on $\text{A}p=x-\beta$, the probability of which will be $\frac{n(x-\beta)^{n-1}}{a^n}$. And in the same way, taking also into account the chance of the points falling on PB, and proceeding as before, we get

$$\delta=n\int_0^x \frac{(x-\beta)^{n-1}}{a^n}\beta d\beta+n\int_0^y \frac{(y-\beta)^{n-1}}{a^n}\beta d\beta.$$

Again, for the distance Δ', suppose $p\text{P}=\text{P}p'=\beta$ to be the distance of the most remote extreme point from P when the points fall on both sides. Then, the remotest extreme point being at p, the other $n-1$ points must fall on $pp'=2\beta$, but not on $p\text{P}=\beta$; and if, on the other hand, p' be taken as the remotest extreme point, the other $n-1$ points must fall on $pp'=2\beta$, but not on $\text{P}p'=\beta$. Therefore, if PQ be taken equal to AP, the value of Δ' as regards the chance of the n points falling on AQ will be $2n\int_0^x \frac{(2\beta)^{n-1}-(\beta)^{n-1}}{a^n}\beta d\beta$. If the furthest extreme point p fall between Q and B, the other n—1 points should fall on $\text{A}p=x+\beta$, but not on $\text{P}p=\beta$. We have evidently, therefore,

$$\Delta'=2n\int_0^x \frac{(2\beta)^{n-1}-(\beta)^{n-1}}{a^n}\beta d\beta+n\int_x^y \frac{(x+\beta)^{n-1}-(\beta)^{n-1}}{a^n}\beta d\beta.$$

Lastly, for the distance δ', if the point p in AP be the nearest extreme point from P when the n points are distributed on both sides of P, the remaining $n-1$ points must be on $p\text{B}=y+\beta$, but not on $pp'=2\beta$; and if p' be the nearest extreme point, the remaining points must be found on $\text{A}p'=x+\beta$, but not on $pp'=2\beta$. Hence, as before, we obtain

$$\delta'=n\int_0^x \frac{(x+\beta)^{n-1}-(2\beta)^{n-1}}{a^n}\beta d\beta+n\int_0^x \frac{(y+\beta)^{n-1}-(2\beta)^{n-1}}{a^n}\beta d\beta.$$

After calculating the several integrals, between the prescribed limits, with respect to the variable β, the formulæ we have obtained for Δ, δ, Δ', δ', give the following results:—

$$\left.\begin{aligned}\Delta&=\frac{n(x^{n+1}+y^{n+1})}{(n+1)a^n},\\ \delta&=\frac{x^{n+1}+y^{n+1}}{(n+1)a^n},\\ \Delta'&=\frac{x(2x)^n}{(n+1)a^n}+\frac{n}{n+1}a-x-\Delta,\\ \delta'&=-\frac{x(2x)^n}{(n+1)a^n}+\frac{n}{n+1}a-y+\delta.\end{aligned}\right\}\ \ldots\ (\mathrm{A})$$

These would be the values if the position of the station P were fixed or known. Now, if we suppose every position to be equally probable, the probability of P falling between x and $x+dx$ will be $\frac{dx}{a}$, and, as P may be situated on each half of the line AB, the integrals with respect to x from 0 to $\frac{1}{2}a$ must evidently be doubled. Therefore, when every position of P on AB is equally probable, the true average values of the several distances, subject to the chance of the points falling in the manner described, will be

$$\left.\begin{aligned}\Delta&=2\int_0^{\frac{1}{2}a}\Delta\frac{dx}{a}=\frac{2n}{(n+1)(n+2)}a;\\ \delta&=2\int_0^{\frac{1}{2}a}\delta\frac{dx}{a}=\frac{2}{(n+1)(n+2)}a,\\ \Delta'&=2\int_0^{\frac{1}{2}a}\Delta'\frac{dx}{a}=\frac{3n(n-1)}{4(n+1)(n+2)}a,\\ \delta'&=2\int_0^{\frac{1}{2}a}\delta'\frac{dx}{a}=\frac{n(n-1)}{4(n+1)(n+2)}a;\end{aligned}\right\}\ \ldots\ (\mathrm{B})$$

and these are, in fact, the average values that may be fairly apportioned to each separate trial.

Now, when the points fall on the same side of P, the piecer will simply have to walk the distance Δ to the furthest point; and when the points fall on both sides of P he will have to proceed first to the nearest extreme point before going to the furthest point, and therefore in this case will walk over the distance $2\delta'+\Delta'$. Hence the average distance he may be expected to walk, including the chances of all circumstances, will be

$$\mathrm{D}=\Delta+\Delta'+2\delta'=\frac{n(3+5n)}{4(n+1)(n+2)}a,$$

which is particularly neat and simple as regards numerical calculation.

Cor. 1. Give to n the consecutive values 1 to 10, &c., and the corresponding average distances severally travelled, when expressed in fractional parts of the line AB, will be respectively $\frac{1}{3}$, $\frac{13}{24}$, $\frac{27}{40}$, $\frac{23}{30}$, $\frac{5}{6}$, $\frac{99}{112}$, $\frac{133}{144}$, $\frac{43}{45}$, $\frac{54}{55}$, $\frac{265}{264}$, &c. When the number of points is greater than 9, the distance D is greater than the whole line AB. Therefore, when this is the case, the piecer would do well to walk backwards and forwards alternately the entire distance AB.

Cor. 2. The average value of the distance or separation of the two extreme points is

$$(\Delta-\delta)+(\Delta'+\delta')=\frac{n-1}{n+1}a,$$

which is the same as if the n points divided the line AB into $n+1$ equal parts. Also $\Delta'=3\delta'$ and $\Delta=n\delta$.

Cor. 3. The chance of the n points falling wholly on one side of $P=\frac{x^n+y^n}{a^n}$; and, therefore, under all positions of P, this chance $=2\int_0^{\frac{1}{2}a}\frac{dx}{a}\cdot\frac{x^n+y^n}{a^n}=\frac{2}{n+1}$. Taking this from unity we have the chance of the points occupying both sides of $P=\frac{n-1}{n+1}$.

Hence, dividing the foregoing contingent values of Δ, δ, and Δ', δ', by these chances, we have for the average distances, irrespective of contingency, the following values:—

$$\Delta=\frac{n}{n+2}a,\ \delta=\frac{1}{n+2}a,\ \Delta'=\frac{3n}{4(n+2)}a,\ \delta'=\frac{n}{4(n+2)}a.$$

These are the average values that might be uniformly substituted in place of the distances that actually occur.

Cor. 4. When the initial position of P is known, the values of the contingent distances Δ, δ, Δ', δ', are given according to the expressions in equations (A); and taking $\Delta+\Delta'+2\delta'$, these give

$$D=\frac{n-2}{n+1}+x+2\frac{x^{n+1}+y^{n+1}}{n+1}-\frac{x(2x)^n}{n+1}.$$

Therefore for the several values of n we have

$$n=1,\ D=x+(1-x)^2-\tfrac{1}{2};$$

$$n=2,\ D=x+\frac{2\{(1-x)^3-x^3\}}{3};$$

$$n=3,\ D=x+\frac{1+2\{(1-x)^4-3x^4\}}{4};$$

$$n=4,\ \mathrm{D}=x+\frac{2+2\{(1-x)^5-7x^5\}}{5};$$

$$n=5,\ \mathrm{D}=x+\frac{3+2\{(1-x)^6-15x^6\}}{6};$$

$$n=6,\ \mathrm{D}=x+\frac{4+2\{(1-x)^7-31x^7\}}{7};$$

$$n=7,\ \mathrm{D}=x+\frac{5+2\{(1-x)^8-63x^8\}}{8};$$

$$n=8,\ \mathrm{D}=x+\frac{6+2\{(1-x)^9-127x^9\}}{9};$$

$$n=9,\ \mathrm{D}=x+\frac{7+2\{(1-x)^{10}-255x^{10}\}}{10};$$

$$n=10,\ \mathrm{D}=x+\frac{8+2\{(1-x)^{11}-511x^{11}\}}{11};$$

&c.

For $x=0$, ·1, ·2, ·3, ·4, ·5, the results are shown in the following table, and an additional column is given, showing the corresponding general mean values deduced in Cor. 1.

Values of D *for given positions of* P *on* AB.

n	Distance AP=x, in tenths of the line AB.						General Mean when x is unknown.
	·0	·1	·2	·3	·4	·5	
1	·5000	·4100	·3400	·2900	·2600	·2500	·3333
2	·6667	·5853	·5360	·5107	·5013	·5000	·5417
3	·7500	·6779	·6524	·6579	·6764	·6875	·6750
4	·8000	·7362	·7302	·7604	·8024	·8260	·7667
5	·8333	·7771	·7870	·8356	·8951	·9271	·8333
6	·8571	·8081	·8312	·8933	·9649	1·0045	·8839
7	·8750	·8326	·8669	·9384	1·0189	1·0645	·9236
8	·8889	·8528	·8965	·9751	1·0615	1·1120	·9556
9	·9000	·8697	·9215	1·0053	1·0959	1·1504	·9818
10	·9091	·8843	·9429	1·0307	1·1240	1·1820	1·0038

It will be observed that the most favourable position, or that in which the expected distance D is a minimum, when $n=1$, is in the middle of the line; but as n increases it gradually approaches the extremity. It is also remarkable that the most unfavourable position, or that in which D is a maximum, is situated at the extremity of the line when $n=1$, 2, 3; and becomes suddenly transferred to the centre for all higher values of n.